

# 2.4

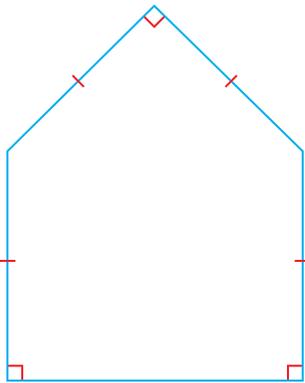
## Angle Properties in Polygons

### YOU WILL NEED

- dynamic geometry software  
OR protractor and ruler

### EXPLORE...

- A pentagon has three right angles and four sides of equal length, as shown. What is the sum of the measures of the angles in the pentagon?



### GOAL

Determine properties of angles in polygons, and use these properties to solve problems.

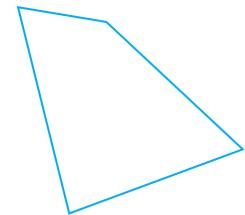
### INVESTIGATE the Math

In Lesson 2.3, you proved properties involving the interior and exterior angles of triangles. You can use these properties to develop general relationships involving the interior and exterior angles of polygons.

- ?** How is the number of sides in a polygon related to the sum of its interior angles and the sum of its exterior angles?

### Part 1 Interior Angles

- A.** Giuseppe says that he can determine the sum of the measures of the interior angles of this quadrilateral by including the diagonals in the diagram. Is he correct? Explain.
- B.** Determine the sum of the measures of the interior angles of any quadrilateral.
- C.** Draw the polygons listed in the table below. Create triangles to help you determine the sum of the measures of their interior angles. Record your results in a table like the one below.

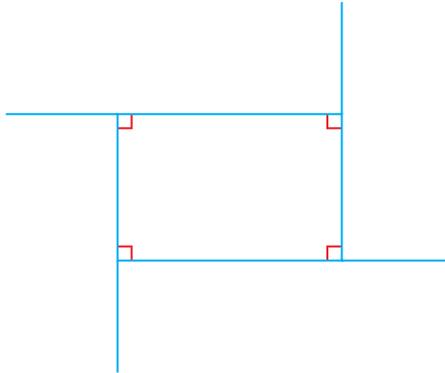


Polygon	Number of Sides	Number of Triangles	Sum of Angle Measures
triangle	3	1	$180^\circ$
quadrilateral	4		
pentagon	5		
hexagon	6		
heptagon	7		
octagon	8		

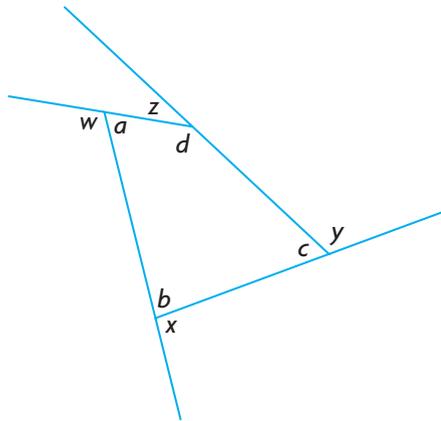
- D.** Make a conjecture about the relationship between the sum of the measures of the interior angles of a polygon,  $S$ , and the number of sides of the polygon,  $n$ .
- E.** Use your conjecture to predict the sum of the measures of the interior angles of a dodecagon (12 sides). Verify your prediction using triangles.

## Part 2 Exterior Angles

- F. Draw a rectangle. Extend each side of the rectangle so that the rectangle has one exterior angle for each interior angle. Determine the sum of the measures of the exterior angles.



- G. What do you notice about the sum of the measures of each exterior angle of your rectangle and its adjacent interior angle? Would this relationship also hold for the exterior and interior angles of the irregular quadrilateral shown? Explain.



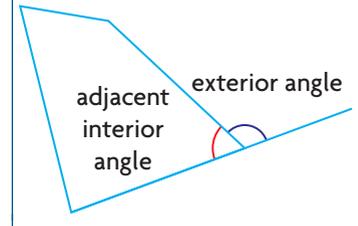
- H. Make a conjecture about the sum of the measures of the exterior angles of any quadrilateral. Test your conjecture.
- I. Draw a pentagon. Extend each side of the pentagon so that the pentagon has one exterior angle for each interior angle. Based on your diagram, revise your conjecture to include pentagons. Test your revised conjecture.
- J. Do you think your revised conjecture will hold for polygons that have more than five sides? Explain and verify by testing.

## Reflecting

- K. Compare your results for the sums of the measures of the interior angles of polygons with your classmates' results. Do you think your conjecture from part D will be true for any polygon? Explain.
- L. Compare your results for the sums of the measures of the exterior angles of polygons with your classmates' results. Do you think your conjecture from part I will apply to any polygon? Explain.

### Communication **Tip**

When a side of a polygon is extended, two angles are created. The angle that is considered to be the exterior angle is adjacent to the interior angle at the vertex.



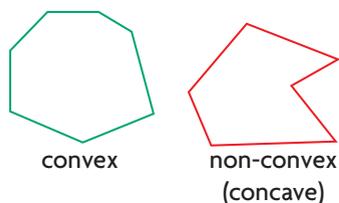
## APPLY the Math

### EXAMPLE 1

### Reasoning about the sum of the interior angles of a polygon

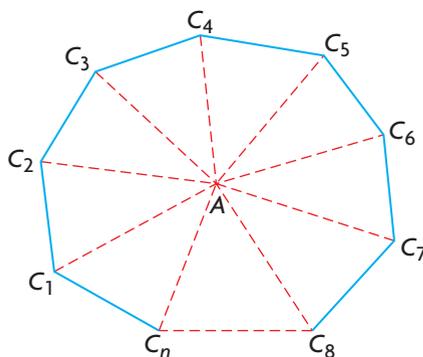
#### convex polygon

A polygon in which each interior angle measures less than  $180^\circ$ .



Prove that the sum of the measures of the interior angles of any  $n$ -sided **convex polygon** can be expressed as  $180^\circ(n - 2)$ .

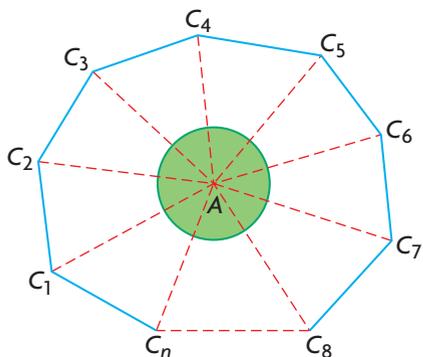
#### Viktor's Solution



I drew an  $n$ -sided polygon. I represented the  $n$ th side using a broken line. I selected a point in the interior of the polygon and then drew line segments from this point to each vertex of the polygon. The polygon is now separated into  $n$  triangles.

The sum of the measures of the angles in each triangle is  $180^\circ$ .

The sum of the measures of the angles in  $n$  triangles is  $n(180^\circ)$ .



Two angles in each triangle combine with angles in the adjacent triangles to form two interior angles of the polygon.

Each triangle also has an angle at vertex  $A$ . The sum of the measures of the angles at  $A$  is  $360^\circ$  because these angles make up a complete rotation. These angles do not contribute to the sum of the interior angles of the polygon.

The sum of the measures of the interior angles of the polygon,  $S(n)$ , where  $n$  is the number of sides of the polygon, can be expressed as:

$$S(n) = 180^\circ n - 360^\circ$$

$$S(n) = 180^\circ(n - 2)$$

The sum of the measures of the interior angles of a convex polygon can be expressed as  $180^\circ(n - 2)$ .

#### Your Turn

Explain why Viktor's solution cannot be used to show whether the expression  $180^\circ(n - 2)$  applies to non-convex polygons.

**EXAMPLE 2****Reasoning about angles in a regular polygon**

Outdoor furniture and structures like gazebos sometimes use a regular hexagon in their building plan. Determine the measure of each interior angle of a regular hexagon.

**Nazra's Solution**

Let  $S(n)$  represent the sum of the measures of the interior angles of the polygon, where  $n$  is the number of sides of the polygon.

$$S(n) = 180^\circ(n - 2)$$

$$S(6) = 180^\circ[(6) - 2]$$

$$S(6) = 720^\circ$$

$$\frac{720^\circ}{6} = 120^\circ$$

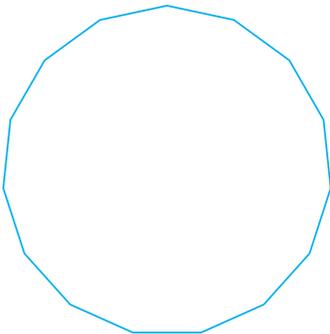
The measure of each interior angle of a regular hexagon is  $120^\circ$ .

A hexagon has six sides, so  $n = 6$ .

Since the measures of the angles in a regular hexagon are equal, each angle must measure  $\frac{1}{6}$  of the sum of the angles.

**Your Turn**

Determine the measure of each interior angle of a regular 15-sided polygon (a pentadecagon).



**EXAMPLE 3****Visualizing tessellations**

A floor tiler designs custom floors using tiles in the shape of regular polygons. Can the tiler use congruent regular octagons and congruent squares to tile a floor, if they have the same side length?

**Vanessa's Solution**

$$S(n) = 180^\circ(n - 2)$$

$$S(8) = 180^\circ[(8) - 2]$$

$$S(8) = 1080^\circ$$

$$\frac{1080^\circ}{8} = 135^\circ$$

The measure of each interior angle in a regular octagon is  $135^\circ$ .

The measure of each internal angle in a square is  $90^\circ$ .

Two octagons fit together, forming an angle that measures:

$$2(135^\circ) = 270^\circ.$$

This leaves a gap of  $90^\circ$ .

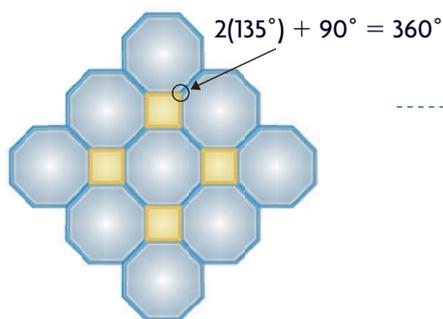
$$2(135^\circ) + 90^\circ = 360^\circ$$

A square can fit in this gap if the sides of the square are the same length as the sides of the octagon.

Since an octagon has eight sides,  $n = 8$ .

First, I determined the sum of the measures of the interior angles of an octagon. Then I determined the measure of each interior angle in a regular octagon.

I knew that three octagons would not fit together, as the sum of the angles would be greater than  $360^\circ$ .



I drew what I had visualized using dynamic geometry software.

The tiler can tile a floor using regular octagons and squares when the polygons have the same side length.

**Your Turn**

Can a tiling pattern be created using regular hexagons and equilateral triangles that have the same side length? Explain.

## In Summary

### Key Idea

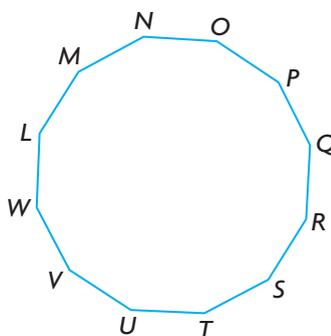
- You can prove properties of angles in polygons using other angle properties that have already been proved.

### Need to Know

- The sum of the measures of the interior angles of a convex polygon with  $n$  sides can be expressed as  $180^\circ(n - 2)$ .
- The measure of each interior angle of a regular polygon is  $\frac{180^\circ(n - 2)}{n}$ .
- The sum of the measures of the exterior angles of any convex polygon is  $360^\circ$ .

## CHECK Your Understanding

- Determine the sum of the measures of the interior angles of a regular dodecagon.
  - Determine the measure of each interior angle of a regular dodecagon.



- Determine the sum of the measures of the angles in a 20-sided convex polygon.
- The sum of the measures of the interior angles of an unknown polygon is  $3060^\circ$ . Determine the number of sides that the polygon has.

## PRACTISING

- Honeybees make honeycombs to store their honey. The base of each honeycomb is roughly a regular hexagon. Explain why a regular hexagon can be used to tile a surface.



5. Is it possible to create a tiling pattern with parallelograms? Explain.
6. Determine the measure of each interior angle of a loonie.



7. Each interior angle of a regular convex polygon measures  $140^\circ$ .
  - a) Prove that the polygon has nine sides.
  - b) Verify that the sum of the measures of the exterior angles is  $360^\circ$ .
8.
  - a) Determine the measure of each exterior angle of a regular octagon.
  - b) Use your answer for part a) to determine the measure of each interior angle of a regular octagon.
  - c) Use your answer for part b) to determine the sum of the interior angles of a regular octagon.
  - d) Use the function
 
$$S(n) = 180^\circ(n - 2)$$
 to determine the sum of the interior angles of a regular octagon. Compare your answer with the sum you determined in part c).
9.
  - a) Wallace claims that the opposite sides in any regular hexagon are parallel. Do you agree or disagree? Justify your decision.
  - b) Make a conjecture about parallel sides in regular polygons.

## Math in Action

### “Circular” Homes

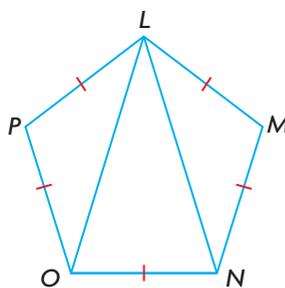
A building based on a circular floor plan has about 11% less outdoor wall surface area than one based on a square floor plan of the same area. This means less heat is lost through the walls in winter, lowering utility bills.

Most “circular” buildings actually use regular polygons for their floor plans.

- Determine the exterior angle measures of a floor plan that is a regular polygon with each of the following number of sides: 12, 18, 24. Explain why a building would be closer to circular as the number of sides increases.
- List some practical limitations on the number of sides a building could have.
- Based on the practical limitations, suggest an optimal number of sides for a home. Sketch a floor plan for a home with this number of sides.



10.  $LMNOP$  is a regular pentagon.
- Determine the measure of  $\angle OLN$ .
  - What kind of triangle is  $\triangle LON$ ? Explain how you know.

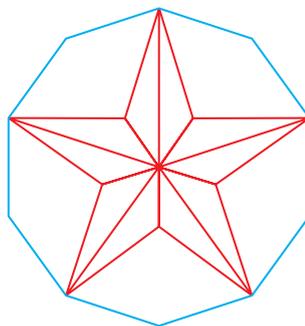


11. Sandy designed this logo for the jerseys worn by her softball team. She told the graphic artist that each interior angle of the regular decagon should measure  $162^\circ$ , based on this calculation:

$$S(10) = \frac{180^\circ(10 - 1)}{10}$$

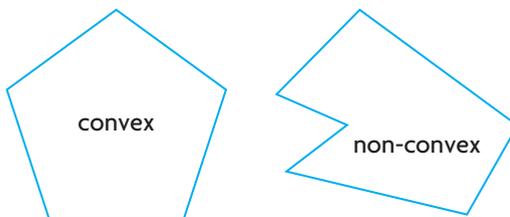
$$S(10) = \frac{1620^\circ}{10}$$

$$S(10) = 162^\circ$$



Identify the error she made and determine the correct angle.

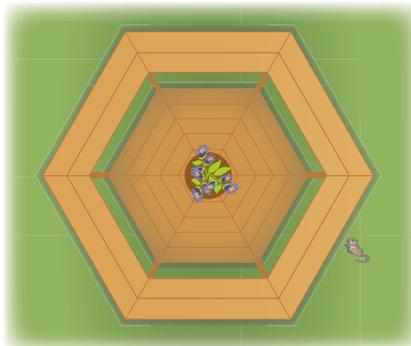
12. Astrid claims that drawing lines through a polygon can be used as a test to determine whether the polygon is convex or non-convex (concave).



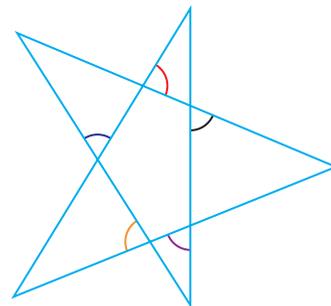
- Describe a test that involves drawing a single line.
- Describe a test that involves drawing diagonals.

13. Martin is planning to build a hexagonal picnic table, as shown.

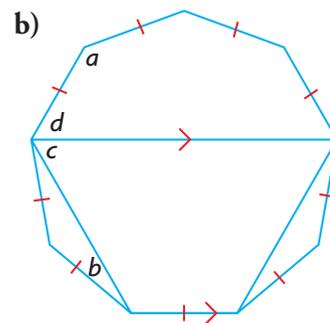
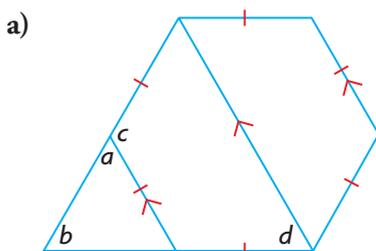
- Determine the angles at the ends of each piece of wood that Martin needs to cut for the seats.
- How would these angles change if Martin decided to make an octagonal table instead?



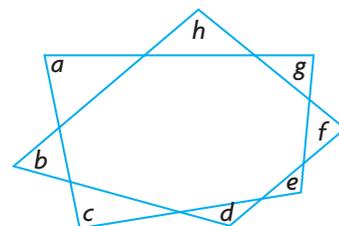
14. Three exterior angles of a convex pentagon measure  $70^\circ$ ,  $60^\circ$ , and  $90^\circ$ . The other two exterior angles are congruent. Determine the measures of the interior angles of the pentagon.
15. Determine the sum of the measures of the indicated angles.



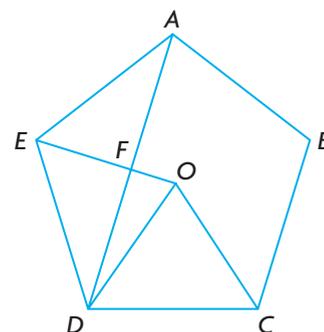
16. In each figure, the congruent sides form a regular polygon. Determine the values of  $a$ ,  $b$ ,  $c$ , and  $d$ .



17. Determine the sum of the measures of the indicated angles.



18. Given:  $ABCDE$  is a regular pentagon with centre  $O$ .  
 $\triangle EOD$  is isosceles, with  $EO = DO$ .  
 $DO = CO$   
 Prove:  $\triangle EFD$  is a right triangle.



## Closing

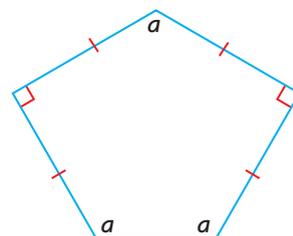
19. The function representing the sum of the measures of the interior angles of a polygon with  $n$  sides is:

$$S(n) = 180^\circ(n - 2)$$

Explain how the expression on the right can be deduced by considering a polygon with  $n$  sides.

## Extending

20. A pentagon tile has two  $90^\circ$  angles. The other three angles are equal. Is it possible to create a tiling pattern using only this tile? Justify your answer.
21. Each interior angle of a regular polygon is five times as large as its corresponding exterior angle. What is the common name of this polygon?



## History | Connection

### Buckyballs—Polygons in 3-D

Richard Buckminster “Bucky” Fuller (1895–1983) was an American architect and inventor who spent time working in Canada. He developed the geodesic dome and built a famous example, now called the Montréal Biosphere, for Expo 1967. A spin-off from Fuller’s dome design was the buckyball, which became the official design for the soccer ball used in the 1970 World Cup.

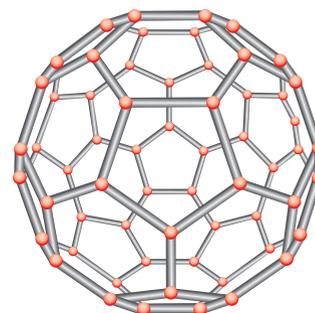
In 1985, scientists discovered carbon molecules that resembled Fuller’s geodesic sphere. These molecules were named fullerenes, after Fuller.



The Montréal Biosphere and its architect



FIFA soccer ball, 1970



Carbon molecule, C<sub>60</sub>

- A. Identify the polygons that were used to create the buckyball.
- B. Predict the sum of the three interior angles at each vertex of the buckyball. Check your prediction.
- C. Explain why the value you found in part B makes sense.