1.4 Proving Conjectures: Deductive Reasoning

GOAL

Prove mathematical statements using a logical argument.

YOU WILL NEED

- calculator
- ruler

EXPLORE…

- How can the conjecture "All teens like music" be supported inductively? Can this conjecture be proved? Explain.

proof

A mathematical argument showing that a statement is valid in all cases, or that no counterexample exists.

generalization

A principle, statement, or idea that has general application.

LEARN ABOUT the Math

Jon discovered a pattern when adding integers:

$$1 + 2 + 3 + 4 + 5 = 15$$
$$(-15) + (-14) + (-13) + (-12) + (-11) = -65$$
$$(-3) + (-2) + (-1) + 0 + 1 = -5$$

He claims that whenever you add five consecutive integers, the sum is always 5 times the median of the numbers.

? How can you prove that Jon's conjecture is true for all integers?

EXAMPLE 1 Connecting conjectures with reasoning

Prove that Jon's conjecture is true for all integers.

Pat's Solution

$$5(3) = 15$$
$$5(-13) = -65$$
$$5(-1) = -5$$

The median is the middle number in a set of integers when the integers are arranged in consecutive order. I observed that Jon's conjecture was true in each of his examples.

$$210 + 211 + 212 + 213 + 214 = 1060$$
$$5(212) = 1060$$

I tried a sample with greater integers, and the conjecture still worked.

Let x represent any integer.
Let S represent the sum of five consecutive integers.
$S = (x - 2) + (x - 1) + x + (x + 1) + (x + 2)$

I decided to start my **proof** by representing the sum of five consecutive integers. I chose x as the median and then wrote a **generalization** for the sum.

$S = (x + x + x + x + x) + (-2 + (-1) + 0 + 1 + 2)$
$S = 5x + 0$

I simplified by gathering like terms.

$S = 5x$
Jon's conjecture is true for all integers.

Since x represents the median of five consecutive integers, $5x$ will always represent the sum.

deductive reasoning

Drawing a specific conclusion through logical reasoning by starting with general assumptions that are known to be valid.

Reflecting

A. What type of reasoning did Jon use to make his conjecture?

B. Pat used **deductive reasoning** to prove Jon's conjecture. How does this differ from the type of reasoning that Jon used?

APPLY the Math

EXAMPLE 2 Using deductive reasoning to generalize a conjecture

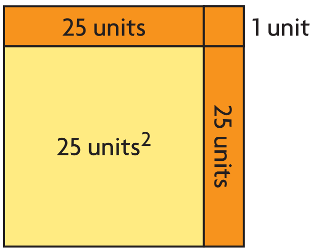
In Lesson 1.3, page 19, Luke found more support for Steffan's conjecture from Lesson 1.1, page 9—that the difference between consecutive perfect squares is always an odd number.

Determine the general case to prove Steffan's conjecture.

Gord's Solution

The difference between consecutive perfect squares is always an odd number.

Steffan's conjecture has worked for consecutive perfect squares with sides of 1 to 7 units.



$26^2 - 25^2 = 2(25) + 1$

$26^2 - 25^2 = 51$

I tried a sample using even greater squares: 26^2 and 25^2.

The difference is the two sets of 25 unit tiles, plus a single unit tile.

Let x be any natural number.
Let D be the difference between consecutive perfect squares.

$D = (x + 1)^2 - x^2$

Since the conjecture has been supported with specific examples, I decided to express the conjecture as a general statement. I chose x to be the length of the smaller square's sides. The larger square's sides would then be $x + 1$.

$D = x^2 + x + x + 1 - x^2$

$D = x^2 + 2x + 1 - x^2$

$D = 2x + 1$

I expanded and simplified my expression. Since x represents any natural number, $2x$ is an even number, and $2x + 1$ is an odd number.

Steffan's conjecture, that the difference of consecutive perfect squares is always an odd number, has been proved for all natural numbers.

Your Turn

In Lesson 1.3, Luke visualized the generalization but did not develop the reasoning to support it. How did the visualization explained by Luke help Gord develop the general statement? Explain.

EXAMPLE 3 Using deductive reasoning to make a valid conclusion

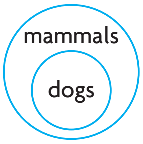
All dogs are mammals. All mammals are vertebrates. Shaggy is a dog.

What can be deduced about Shaggy?

Oscar's Solution

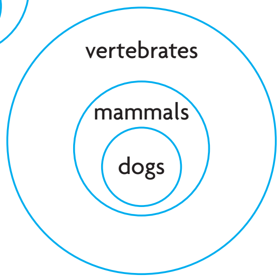
Shaggy is a dog.

All dogs are mammals.



These statements are given. I represented them using a Venn diagram.

All mammals are vertebrates.



This statement is given. I modified my diagram.

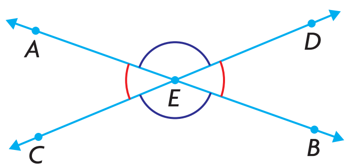
Therefore, through deductive reasoning, Shaggy is a mammal and a vertebrate.

Your Turn

Weight-lifting builds muscle. Muscle makes you strong. Strength improves balance. Inez lifts weights. What can be deduced about Inez?

EXAMPLE 4 Using deductive reasoning to prove a geometric conjecture

Prove that when two straight lines intersect, the vertically opposite angles are equal.



Jose's Solution: Reasoning in a two-column proof

Statement	Justification
$\angle AEC + \angle AED = 180°$	Supplementary angles
$\angle AEC = 180° - \angle AED$	Subtraction property
$\angle BED + \angle AED = 180°$	Supplementary angles
$\angle BED = 180° - \angle AED$	Subtraction property
$\angle AEC = \angle BED$	**Transitive property**

Your Turn

Use a **two-column proof** to prove that $\angle AED$ and $\angle CEB$ are equal.

transitive property

If two quantities are equal to the same quantity, then they are equal to each other.
If $a = b$ and $b = c$, then $a = c$.

two-column proof

A presentation of a logical argument involving deductive reasoning in which the statements of the argument are written in one column and the justifications for the statements are written in the other column.

EXAMPLE 5 Communicating reasoning about a divisibility rule

The following rule can be used to determine whether a number is divisible by 3:

Add the digits, and determine if the sum is divisible by 3. If the sum is divisible by 3, then the original number is divisible by 3.

Use deductive reasoning to prove that the divisibility rule for 3 is valid for two-digit numbers.

Lee's Solution

Expanded Number Forms		
Number	**Expanded Form (Words)**	**Expanded Form (Numbers)**
9	9 ones	9(1)
27	2 tens and 7 ones	2(10) + 7(1)
729	7 hundreds and 2 tens and 9 ones	7(100) + 2(10) + 9(1)
ab	*a* tens and *b* ones	*a*(10) + *b*(1)

Let *ab* represent any two-digit number.

I let *ab* represent any two-digit number.

$ab = 10a + b$

Since any number can be written in expanded form, I wrote *ab* in expanded form.

$ab = (9a + 1a) + b$
$ab = 9a + (a + b)$

I decomposed 10*a* into an equivalent sum. I used 9*a* because I knew that 9*a* is divisible by 3, since 3 is a factor of 9.

The number *ab* is divisible by 3 only when $(a + b)$ is divisible by 3.

From this equivalent expression, I concluded that *ab* is divisible by 3 only when both 9*a* and $(a + b)$ are divisible by 3. I knew that 9*a* is always divisible by 3, so I concluded that *ab* is divisible by 3 only when $(a + b)$ is divisible by 3.

The divisibility rule has been proved for two-digit numbers.

Your Turn

Use similar reasoning to prove that the divisibility rule for 3 is valid for three-digit numbers.

In Summary

Key Idea

- Deductive reasoning involves starting with general assumptions that are known to be true and, through logical reasoning, arriving at a specific conclusion.

Need to Know

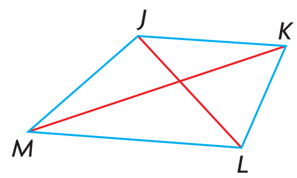
- A conjecture has been proved only when it has been shown to be true for every possible case or example. This is accomplished by creating a proof that involves general cases.
- When you apply the principles of deductive reasoning correctly, you can be sure that the conclusion you draw is valid.
- The transitive property is often useful in deductive reasoning. It can be stated as follows: Things that are equal to the same thing are equal to each other. If $a = b$ and $b = c$, then $a = c$.
- A demonstration using an example is *not* a proof.

CHECK Your Understanding

1. Chuck made the conjecture that the sum of any seven consecutive integers is 7 times the median. Prove Chuck's conjecture.

2. Jim is a barber. Everyone whose hair is cut by Jim gets a good haircut. Austin's hair was cut by Jim. What can you deduce about Austin?

3. Lila drew a quadrilateral and its diagonals. What could Lila deduce about the angles formed at the intersection of the diagonals?

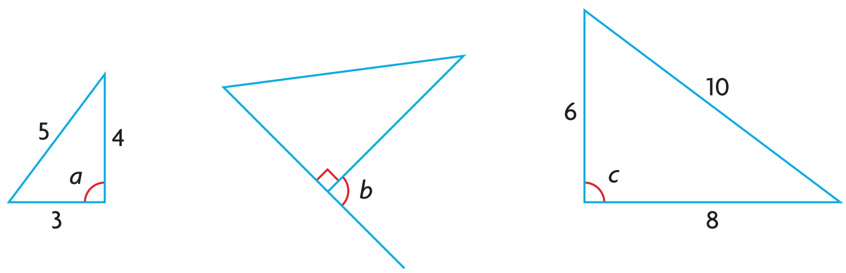


PRACTISING

4. Prove that the sum of two even integers is always even.

5. Prove that the product of an even integer and an odd integer is always even.

6. Prove that a, b, and c are equal.



7. Drew created this step-by-step number trick:
- Choose any number.
- Multiply by 4.
- Add 10.
- Divide by 2.
- Subtract 5.
- Divide by 2.
- Add 3.

a) Show inductively, using three examples, that the result is always 3 more than the chosen number.

b) Prove deductively that the result is always 3 more than the chosen number.

8. Examine the following example of deductive reasoning. Why is it faulty?

Given: Khaki pants are comfortable. Comfortable pants are expensive. Adrian's pants are not khaki pants.
Deduction: Adrian's pants are not expensive.

9. Recall Jarrod's number trick from Lesson 1.3, page 24:
- Choose a number.
- Double it.
- Add 6.
- Double again.
- Subtract 4.
- Divide by 4.
- Subtract 2.

Prove that any number you choose will be the final result.

10. Prove that whenever you square an odd integer, the result is odd.

11. Cleo noticed that whenever she determined the difference between the squares of consecutive even numbers or the difference between the squares of consecutive odd numbers, the result was a multiple of 4. Show inductively that this pattern exists. Then prove deductively that it exists.

12. Create a number trick with five or more steps, similar to the number trick in question 9. Your number trick must always result in a final answer of 6. Prove that your number trick will always work.

13. Prove that any four-digit number is divisible by 2 when the last digit in the number is divisible by 2.

14. Prove that any two-digit or three-digit number is divisible by 5 when the last digit in the number is divisible by 5.

15. To determine if a number is divisible by 9, add all the digits of the number and determine if the sum is divisible by 9. If it is, then the number is divisible by 9. Prove that the divisibility rule for 9 works for all two-digit and three-digit numbers.

16. Look for a pattern when any odd number is squared and then divided by 4. Make a conjecture, and then prove your conjecture.

Closing

17. Simon made the following conjecture: When you add three consecutive numbers, your answer is always a multiple of 3. Joan, Garnet, and Jamie took turns presenting their work to prove Simon's conjecture. Which student had the strongest proof? Explain.

Joan's Work	Garnet's Work	Jamie's Work
$1 + 2 + 3 = 6$ $3 \cdot 2 = 6$ $2 + 3 + 4 = 9$ $3 \cdot 3 = 9$ $3 + 4 + 5 = 12$ $3 \cdot 4 = 12$ $4 + 5 + 6 = 15$ $3 \cdot 5 = 15$ $5 + 6 + 7 = 18$ $3 \cdot 6 = 18$ and so on ... Simon's conjecture is valid.	$3 + 4 + 5$ The two outside numbers (3 and 5) add to give twice the middle number (4). All three numbers add to give 3 times the middle number. Simon's conjecture is valid.	Let the numbers be n, $n + 1$, and $n + 2$. $n + n + 1 + n + 2 = 3n + 3$ $n + n + 1 + n + 2 = 3(n + 1)$ Simon's conjecture is valid.

Extending

18. The table below outlines one possible personal strategy for calculating the square of a number.

Step	Method	Example
1	Round the number down to the nearest multiple of 10.	37 is the number to be squared. Round down to 30.
2	Determine the difference between the original number and the rounded number. Add the difference to the original number.	$37 - 30 = 7$ $7 + 37 = 44$
3	Multiply the rounded number by the number from step 2.	$(30)(44) = 1320$
4	Add the square of the difference between the original number and the rounded number.	$1320 + 7^2 = 1369$

From the given example, determine deductively the general rule for x^2.

19. Prove that the expression $n^2 + n + 2$ will always generate an even number for every natural number, n.

20. Make a conjecture about the product of two consecutive natural numbers. Prove your conjecture.